# ASYMPTOTIC ANALYSIS OF THE BEHAVIOR OF AN ELASTIC BAR UNDER APERIODIC INTENSIVE LOADING

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UDC 624.074.4

It is established that when aperiodic loads of large intensity act on an elastic bar, the higher modes of stability loss have the highest rates of growth of deflections. A method is indicated for determining the numbers of these modes, when the effect of the longitudinal vibrations on the transverse vibrations is taken into account and when it is not taken into account. A comparison of the results obtained with results of other authors [1-7] is presented.

## 1. Statement of the Problem

The system of equations which takes into account the mutual influence of the longitudinal and transverse vibrations of an inhomogeneous bar has the form

$$(EIw_{,xx})_{,xx} + (EFu_{,x}\, w_{,x})_{,x} + \rho F w_{,tt} = f^*(x, t) \tag{1.1}$$

$$(EFu_{,x})_{,x} = \rho F u_{,tt} \tag{1.2}$$

Here, w and u are the normal and longitudinal displacements of the bar; x and t are the longitudinal coordinate and the time; $E = E(x)$ is Young's modulus; $I = I(x)$ and $F = F(x)$ are the flexural rigidity and the cross-sectional area; $\rho = \rho(x)$ is the density of the material. It is assumed that $E(x)$, $I(x)$, $F(x)$, and $\rho(x)$ are functions which slowly vary along the length of the wave of loss of stability; while $f^*(x, t)$ is a function which is determined by the initial disturbances or imperfections.

We consider a pin-jointed bar of length $l_0$ $(0 \le x \le l_0)$. Let an aperiodic load $N(0, t)$, whose minimum value $\min N(0, t) = N_0$ considerably exceeds the Euler load $P_e$ for the bar, be applied to the bar at rest, for $t = 0$, at the section $x = 0$. Thus, we study the behavior of the bar under intensive loading $N_0/P_e = \eta^2 \gg 1$. The use of asymptotic methods of investigation appears to be natural for problems of this kind.

We assume for the time being that the wave process in the propagation of the longitudinal disturbances can be neglected, i.e., the function $N(x, t)$ is given, and this function is sufficiently smooth. Then, Eq. (1.1) is rewritten in the form

$$(EIw_{,xx})_{,xx} + (Nw_{,x})_{,x} + \rho F w_{,tt} = f^*(x, t) \quad (0 \leqslant x \leqslant l_0) \tag{1.3}$$

The initial and boundary conditions for Eq. (1.3) are

$$w = w_{,t} = 0 \quad (t = 0), \quad w = w_{,xx} = 0 \quad (x = 0, l_0) \tag{1.4}$$

Before proceeding to an asymptotic analysis of the problem (1.3), (1.4), we introduce the following dimensionless parameters and estimate the order of the individual terms, with $N_0/P_e = \eta^2 \gg 1$

$$x_1 = x / l_0, \quad t_1 = \frac{c^* t}{l_0}, \quad c^* = \left(\frac{E^*}{\rho^*}\right)^{1/2}, \quad E^* = \frac{1}{l_0}\int_0^{l_0} E(x)\, dx,$$

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 160-170, May-June, 1972. Original article submitted November 19, 1971.

$$\rho^* = \frac{1}{l_0}\int_0^{l_0}\rho(x)\,dx$$

Here, $c^*$ is the average velocity. In the following the index 1 of the new variables is omitted throughout. The relationship (1.3) in terms of the new variables, if we group the second-degree terms, assumes the form

$$w_{,xxxx} + \eta^2 a(x,t) w_{,xx} + b(x) w_{,tt} + B = f(x,t) \tag{1.5}$$

$$\eta^2 a(x,t) = \frac{Nl_0^2}{EI}, \quad b(x) = \frac{\rho(x)E^* l_0^2}{\rho^* E(x) r^2}, \quad f(x,t) = \frac{l_0^4 f^*(x,t)}{EI} \quad (0 \leqslant x \leqslant 1)$$

Here, $\eta \gg 1$ is a large parameter which characterizes the intensity of loading, $r = r(x)$ is the radius of gyration of the cross section of the bar, and B denotes the second-degree terms of the equation.

The initial and boundary conditions retain the form (1.4).

We estimate the order of the individual terms in (1.5), following [1]. We put

$$a(x,t) = c_1, \; b(x) = c_2, \; B \equiv 0$$

where $c_1$ and $c_2$ are certain constants. Then, the solution of the homogeneous equation (1.5) can be found in the form

$$w^* = A\exp(i\eta\mu_0 x + \eta^2\mu_{00} t) \tag{1.6}$$

if we single out one degree of freedom of the system with distributed parameters. We choose the degree of freedom which corresponds to the maximum of the exponential index (see [1]). Consequently

$$\mu_0^2 = c_1/2, \; \mu_{00}^2 = c_1/4c_2 \tag{1.7}$$

The constants $\eta\mu_0$ and $\eta^2\mu_{00}$ characterize the variability of the solution with respect to the coordinates x and t. We note that the solution (1.6) has a different order of the derivatives with respect to x and t relative to $\eta$

$$\left|\frac{\partial^j w^*}{\partial x^j}\right| = \eta^j O(w^*), \quad \frac{\partial^j w^*}{\partial t^j} = \eta^{2j} O(w^*) \quad (j = 1,2,\ldots) \tag{1.8}$$

## 2. Asymptotic Analysis of a System with One Degree of Freedom

Following the commonly used approaches to asymptotic integration of both ordinary and partial differential equations [8, 9], we seek the solution of the homogeneous equation (1.5), with variable, but weakly varying coefficients, in the form

$$w = Q(t,x,\eta) W(x,t,\eta) \tag{2.1}$$

$$W = z(x,t,\eta)\exp\int_0^x i\eta\mu_1(x,t)\,dx$$

$$Q = C_1 Z(t,x,\eta)\exp\int_0^t [\eta^2\mu_{22}(t,x) + \eta\mu_{21}(t,x)]\,dt$$

Here, $\eta\mu_1$ and $\eta^2\mu_{22} + \eta\mu_{21}$ are the functions characterizing the variability of the solution with respect to x and t, respectively, while z and Z are slowly varying functions, and $C_1$ is an arbitrary constant. For the function $\mu_1(x,t)$ the basic variable is x, while for the functions $\mu_{22}(t,x)$ and $\mu_{21}(t,x)$ the basic variable is t [see (1.6)], i.e.,

$$\frac{\partial^j \mu_1}{\partial t^j} \ll \mu_1 \quad (j = 1,2), \quad \frac{\partial^j \mu_{2k}}{\partial x^j} \ll \mu_{2k} \quad (j = 1,2,3,4, \; k = 1,2) \tag{2.2}$$

The order of variability of the solution (2.1) with respect to x and t for an inhomogeneous bar under aperiodic intensive loading

$$\left|\frac{\partial^j w}{\partial x^j}\right| = \eta^j O(w), \quad \frac{\partial^j w}{\partial t^j} = \eta^{2j} O(w), \quad j = 1,2, \ldots, \quad \eta \gg 1 \tag{2.3}$$

agrees with the variability of the solution with respect to x or t for a homogeneous bar under constant loading [see (1.8)].

The asymptotic form of the solution (2.1) is substituted into the homogeneous equation (1.5). We use the inequalities (2.2) and the relations (2.3). We group the terms with the corresponding powers of the large parameter $\eta$. For example, comparing the terms having the highest power of the large parameter, i.e., those having $\eta^4$, we obtain the following equation:

$$\mu_1^4 - a(x, t)\mu_1^2 + b(x)\mu_{22}^2 = 0 \tag{2.4}$$

The relation (2.4) contains two unknown functions $\mu_1$ and $\mu_{22}$. Just as in the derivation of the solution of an equation with constant coefficients, we stipulate that the function $\mu_1(x, t)$, for arbitrary x and t, results in the maximum of the expression

$$\mu_1^4 - a(x, t)\mu_1^2$$

Then

$$\mu_1^2 = {}^1/_2\, a(x, t) \tag{2.5}$$

From (2.4), we have

$$\mu_{22}^2 = a^2(x, t) \,/\, 4b(x) \tag{2.6}$$

It is easy to see that (2.5) and (2.6) very much remind one of (1.7).

We now proceed to determine the slowly varying function $z(x, t, \eta)$ which characterizes the variation of the amplitude of the mode of loss of stability of the bar in the process of motion. We put

$$z(x, t, \eta) = z_0(x, t) + \eta^{-1} z_1(x, t) + \eta^{-2} z_2(x, t) + \ldots \tag{2.7}$$

In this section, we neglect all terms except the first.

Equation (2.5) determines the mode of loss of stability which increases with the greatest rapidity. For this mode of loss of stability the equation

$$W_{,xxxx} + \eta^2\mu_1^2(x, t) W_{,xx} + B' = 0 \tag{2.8}$$

holds.

Here, B' are the lower terms of the equation.

The boundary conditions are not set up for this equation. We only stipulate that the function be an almost periodic function

$$W(x, t, \eta) = z_0(x, t) \exp\left\{ i\eta \int_0^x \mu_1(x, t)\, dx \right\} \tag{2.9}$$

The second and fourth derivatives of the function W (2.9) are substituted into (2.8), the terms with $\eta^4$ and $\eta^3$ are equated, and two equations are obtained. The first of them is satisfied identically, while the second after transformations assumes the form

$$\frac{dz_0}{z_0} = -\frac{5}{2}\frac{\mu_{1,x}}{\mu_1} dx \tag{2.10}$$

Equation (2.10) is valid if in (2.8), we neglect the lower terms. This equation, when t is a parameter, is integrated by quadrature

$$\ln|z_0| = -\frac{5}{2}\int_0^x \frac{\mu_{1,x}}{\mu_1} dx \tag{2.11}$$

Thus, the distribution of amplitudes of the chosen mode of loss of stability appears as if depending on the local conditions of loading of the elastic construction: the local rigidity and the local intensity of loading. If $\mu_1 = \mu_1(x, t)$, then the distribution of amplitudes of the rapidly oscillating function $W(x, t, \eta)$ depends on the x coordinate and the instant of time t.

Now, in the solution (2.1) it remains for us to determine the factor $Z_1(x, t, \eta)$, which characterizes the variation of the growth rate of the mode of loss of stability already determined, from point to point during the process of motion. The system with distributed parameters is replaced by a system with one degree of freedom [see (2.9) and (2.1)].

The expression (2.1) is substituted into the homogeneous equation (1.5)

$$b\,(t,\,x)Q'' - \eta^4 a\,(t,\,x)Q + B^* = 0. \tag{2.12}$$

Here, $b(t, x)$ and $a(t, x)$ are sufficiently smooth functions, $\eta \gg 1$, and $B^*$ denotes second-degree terms.

Equation (2.12) is an equation of rank 2 (see [8]). After the usual calculations, if $Z(t, x, \eta)$ is represented in the form of an asymptotic series

$$Z\,(t,\,x,\,\eta) = Z_0\,(t,\,x) + \eta^{-1}Z_1\,(t,\,x) + \eta^{-2}Z_2\,(t,\,x) + \ldots$$

for $Q(t, x, \eta)$ the relationship

$$Q\,(t,x,\eta) = C_1\Big\{Z_0\,(t,x)\exp\int_0^t \eta^2\mu_{22}\,(t,x)\,dt + \ldots\Big\}, \quad Z_0 = (b\,/\,a)^{1/4} \tag{2.13}$$

is valid.

The expression (2.14) applies for $Z_0$, when $B^* \equiv 0$ in (2.12).

Finally, after renotation, we obtain the expression (2.1) in which the first term behind the integral sign characterizes the rapid oscillation of the solution along the longitudinal coordinate (x is the basic variable) while the second term characterizes the rate of growth of the deflection (t is the basic variable); the factors $z_0(x, t, \eta)$ and $Z_0(t, x, \eta)$, respectively characterize the distribution of amplitudes of the mode of loss of stability and the intensity of growth of this mode, dependent on the time and the longitudinal coordinate. The constant $C_1$ is chosen from the nonhomogeneous equation when in the right side we have singled out a given form of loss of stability.

We note certain special features of the solution obtained for w [see (2.1)]. The function w, generally speaking, does not satisfy the initial and boundary conditions (1.4) of the problem under consideration. The initial conditions are not satisfied, since components corresponding to the exponential function with a negative index with respect to time and a function bounded with respect to $\eta$ are absent from w. However, for $\eta \gg 1$ and large t these components have a secondary importance. The boundary conditions from (1.4) are not satisfied, because the function $\mu_1$ is determined from (2.5). We recall that the solution w is the basic part of the solution of a system with one degree of freedom. In Section 3, where the system with distributed parameters is replaced by a system with several degrees of freedom, a more exact solution of the problem (1.5), (1.4) is presented. This solution, in contrast to (2.1), satisfies the boundary conditions. The solution (2.1) obtained here correctly reflects the quantitative pattern of the phenomenon: the fully determined mode of loss of stability changes most rapidly, while the mode itself depends on the time.

## 3. Asymptotic Analysis of a System with Several Degrees of Freedom

The expression (2.1) obtained above for the basic part of the solution of a system with one degree of freedom, allows us to proceed to the approximation of a system with distributed parameters by a system with several degrees of freedom, and namely, the number of sign changes of the oscillating component of the solution

$$\sin\Big\{\eta\int_0^x \mu_1(x,\,t^\circ)\,dx\Big\}$$

at a fixed instant $t^\circ$ indicates the number in $m(t^\circ)$ of the mode of loss of stability which most intensively grows at $t = t^\circ$. When t is varied, we obtain a certain function $m(t)$.

Let the function m(t) for $0 \le t \le t_0$ ($t_0$ is a certain constant) run through the integer values $m_1, m_2, \ldots, m_k$ ($k \ge 1$). These integer values are the numbers of the modes of loss of stability which most intensively grow at certain instances of time. Therefore, for the approximation of the system with distributed parameters, we choose modes of loss of stability $W_m(x)$, and the solution of the problem (1.5), (1.4) is represented in the form

$$w = \sum_{j=1}^{k} q_m(t) W_m(x), \quad m = m(j) \quad (k \geqslant 1) \tag{3.1}$$

Here, m is an integer function of an integer argument j, i.e., $m = m(j)$ for $j = 1, 2, \ldots, k$, and W(x) are certain averaged modes of loss of stability. These modes are asymptotic representations of solutions of the following problem concerned with eigenfunctions and eigenvalues (we recall that $\lambda \sim \eta^2$ is a large parameter):

$$W_{,xxxx} + \lambda a^*(x) W_{,xx} + B^\circ = 0, \quad W = W_{,xx} = 0 \quad (x = 0, 1) \tag{3.2}$$

$$a^*(x) = \frac{1}{t_0} \int_0^{t_0} a(x, t)\, dt$$

Here, $B^\circ$ denotes the lower terms of the equation. The boundary conditions of the problem (3.2) may have in fact a more complex form.

After the selection of the appropriate degrees of freedom (modes of loss of stability), we proceed to the determination of the amplitudes $q_m$. The expression (3.1) is substituted into Eq. (1.5) and the initial conditions (1.4); the Bubnov–Galerkin procedure is used. For $q_m$, we obtain the Cauchy problem for the following systems of ordinary differential equations

$$\begin{gathered} q_m'' - \eta^4 \sum_{i=1}^{k} c_{im} q_i + B_m = f_m, \quad q_m(0) = q_m' |0| = 0 \\ c_{mm} \gg c_{im} \quad \text{for} \quad i \neq m, \quad m = m(j) \quad (i, j = 1, 2, \ldots, k) \end{gathered} \tag{3.3}$$

Here, $B_m$ denotes second-degree terms; their relative order is not greater than two, i.e., $B_m \sim \eta^2$ [see (2.2)]. The second-degree terms have such an order when the character of loading and the stiffness characteristics of the bar vary but a little. In the system (3.3) each m equation contains within the higher terms only one derivative of the second order $q_m''$. This is a consequence of the simple boundary conditions of the problem (3.2). If the boundary conditions for the equation of the problem (3.2) differ from the conditions

$$W = W_{,xx} = 0 \quad \text{or} \quad W_{,x} = W_{,xxx} = 0 \quad (x = 0, 1) \tag{3.4}$$

then, a system which is analogous to (3.3) has the form

$$\begin{gathered} \sum_{i=1}^{k} \overset{*}{c}_{im} q_i'' - \eta^4 \sum_{i=1}^{k} c_{im} q_i + B_m = f_m, \quad q_m(0) = q_m'(0) = 0, \quad m = m(j) \\ (i, j = 1, 2, \ldots, k) \\ c_{mm} \gg c_{im} \quad \text{for} \quad i \neq m; \quad \overset{*}{c}_{mm} \gg \overset{*}{c}_{im} \quad \text{for} \quad i \neq m, \quad m \gg 1 \end{gathered} \tag{3.5}$$

We note that the system (3.5) is transformed into a decoupled system, when a homogeneous bar under constant load is being considered, if the boundary conditions have the form (3.4).

The systems (3.3) and (3.5) with variable coefficients contain the natural large parameter $\eta$. The solution of the homogeneous system of ordinary differential equations (3.3) or (3.5) of rank 2 [8] (the highest power of the large parameter is 4) is sought in the form of an asymptotic series. For example, for one of the particular solutions of these homogeneous systems the representation

$$q_m = Z_m \exp \int_0^t [\eta^2 \mu_{22m}(t) + \eta \mu_{21m}(t)]\, dt, \quad Z_m = Z_{0m}(t) + \eta^{-1} Z_{1m}(t) + \ldots \tag{3.6}$$

is valid.

In deriving the solution of the nonhomogeneous system of equations, we use the general method of varying the arbitrary constants.

The representation of the solution of the system (3.4) in the form (3.6) is valid, if for a variation of t in the time interval being considered ($0 \le t \le t_0$) not a single root of the "characteristic polynomial" corresponding to the system (3.4) becomes zero, or the intensity of loading does not vary very much. The coinciding real roots of the characteristic polynomial corresponding to (3.3) do not give rise to complications, since the matrix $c = \|c_{im}\|$ has simple elementary divisors, and, consequently, the form of the asymptotic representation of the solution (3.6) is retained (see [8]). Not one of the roots of the characteristic polynomial corresponding to (3.3) becomes zero, if the functions $c_{mm}$ do not change sign, i.e., $C_{mm}(t) > 0$ for $0 \le t \le t_0$.

For a considerable change in the intensity of loading $c_{mm}(t^*_{mm}) = 0$ for $0 \le t^*_{mm} \le t_0$. In this case, we have to bear in mind the turning points (a certain one of the functions $q_m$ may be transformed from an exponentially increasing function into an oscillating function; for example, the exponential index is greater than zero for $t < t^*_{mm}$, the exponential index is a purely imaginary function for $t > t^*_{mm}$). In the presence of turning points the problem of constructing "through" asymptotic forms arises.

Example. We consider a homogeneous bar under nonuniform longitudinal loading. Let in (1.5)

$$a = 2(1 + \alpha x)^2,\ b = b_0 = \text{const},\ \alpha = \text{const},\ B \equiv 0\ (0 \leqslant x \leqslant 1)$$

Then in the expression (2.1) the functions $\mu_1$, $\mu_{22}$, $z_0$, and $Z_0$ have the form

$$\mu_1 = 1 + \alpha x,\ \mu_{22} = \frac{2(1 + \alpha x)}{2b_0^{1/2}},\ z_0 = \frac{1}{(1 + \alpha x)^{5\alpha^2/2}},\ Z_0 = \frac{(b_0/2)^{1/4}}{(1 + \alpha x)^{1/2}}$$

It is obvious that, dependent on the quantity $\alpha$, the oscillation of the solution with respect to the longitudinal coordinate, the rate of growth of the deflection and the distribution of amplitudes can vary substantially.

## 4. Buckling of a Inhomogeneous Bar under an Impact ("One" Degree of Freedom)

In the preceding two sections, we studied the behavior of the bar with the assumption that the velocity of propagation of the disturbance along the x axis is infinite. We drop this assumption. Let at $t = 0$ an intensive load $N(0, t)$ be applied to bar at the section $x = 0$, i.e.,

$$N = N(0, t),\ x = 0,\ t > 0 \tag{4.1}$$

Here, $N(0, t)$ is a sufficiently smooth function.

The problems (1.2) and (4.1) are problems concerned with the propagation of the boundary effects ($0 \le x \le 1$). The reflection from the support $x = 1$ for the time being is not considered. It is assumed that the solution of this problem for the wave equation (1.2) has already been obtained by some method. Thus, we know the compressive force $N_0(x, t)$ and we know the velocity $c(t)$ ($c(x)$), with which the front of the force is displaced along the bar

$$N_0(x, t) = N(x, t)\ (x < l(t)),\ N_0(x, t) \equiv 0\ (x \geqslant l(t)),\ l(t) = \int_0^t c(\zeta)\, d\zeta \tag{4.2}$$

Here, c is a dimensionless velocity for an inhomogeneous bar.

We note that the function $N_0(x, t)$ has a complicated form; for $x = l$, this function is discontinuous: $N_0(l - \varepsilon, t) \ne N_0(l + \varepsilon, t)$ ($\varepsilon > 0$ is a small positive quantity).

As we know, the bending disturbances given by Eq. (1.1) for $x > l(t)$ are not significant. In addition, with a more exact formulation of the problem we obtain equations of the type of the dynamic equations of the Timoshenko beam. From these equations it follows that the velocity of propagation of the bending disturbances $c_*(t)$ is finite and less than $c(t)$ for any instant of time [$c_*(t) < c(t)$]. Therefore, the study of the equation

$$w_{,xxxx} + \eta^2 a(x, t) w_{,xx} + b(x) w_{,tt} + B = f(x, t) \tag{4.3}$$

is conducted on a variable interval [4], $0 \le x \le l(t)$, just as in [5] (for notation see Section 1). The initial and boundary conditions for Eq. (4.3) are

$$w = w_{,t} = 0 \ (t = 0), \ w = w_{,xx} = 0 \ (x = 0), \ w = w_{,x} = 0 \ (x = l(t)) \tag{4.4}$$

Let $a(x, t)$ and $b(x)$ be functions which only slightly vary along the wave of loss of stability.

We introduce the coordinate transformation [5]

$$x = x, \quad \tau = t - \int_0^x \frac{d\xi}{c(\xi)} \tag{4.5}$$

Here, $\tau$ is the true time of action of the compressive force of high intensity.

Equation (4.3) in terms of the new variables preserves its form (t is replaced by $\tau$), if we neglect the second-degree terms. An estimate of these second-degree terms for a homogeneous bar is presented in Section 1 of [5]. In the case being considered, the principal complications in their estimation do not arise for an inhomogeneous bar with smoothly varying rigidity. The second-degree terms can be neglected, if $\max r(x)/\min L(x) \ll 1$ (here x, r and L are quantities with dimensions; L is the length of the wave of loss of stability

The solution of the transformed equation (4.3) is found in the form

$$w = Q(\tau, x, \eta) W(x, \tau, \eta) \tag{4.6}$$

$$W = z_0(x, \tau) \exp \int_0^x i\eta \mu_1(x, \tau)\, dx, \quad Q = C_1 Z_0(\tau, x) \exp\left\{\int_0^\tau [\eta^2 \mu_{22}(\tau, x) + \right.$$

$$\left. + \eta\mu_{21}(\tau, x)]\, d\tau\right\} + C_2 Z_0(\tau, x) \exp\left\{-\int_0^\tau [\eta^2 \mu_{22}(\tau, x) + \eta\mu_{21}(\tau, x)]\, d\tau\right\} + Q^*(\tau)$$

Here, the functions $\mu_1$, $\mu_{22}$, $\mu_{21}$, $z_0$, and $Z_0$ have the same meaning as in Section 2; $C_1$ and $C_2$ are constants, and $Q^*$ is the particular solution of the nonhomogeneous equation for Q. The last equation is obtained after substitution of (4.6) into the transformed equation (4.3) and after appropriate transformations, if we consider that the mode of loss of stability is already given. The constants $C_1$ and $C_2$ are determined from the initial conditions

$$Q(0, x, \eta) = Q_{,\tau}(0, x, \eta) = 0 \qquad (Q^* = Q_{,\tau}^* = 0 \ \text{for} \ \tau = 0)$$

The solution (4.6) thus set up satisfies the boundary conditions (3.4), just as the solution (2.1).

For the determination of the functions $\mu_1(x, \tau)$, $\mu_{22}(\tau, x)$, $z_0(x, \tau)$, and $Z_0(x, \tau)$ the expressions (2.5), (2.6), (2.11), (2.14), apply, if in the latter, we replace t by $\tau$ (the assumptions with which these expressions have been obtained are retained)

$$\mu_1^2 = \frac{a(x, \tau)}{2}, \ \mu_{22}^2 = \frac{a^2(x, \tau)}{4b(x)}, \ \ln|z_0| = -\frac{5}{2}\int_0^x \frac{\mu_{1,x}\, dx}{\mu_1}, \ Z_0 = \left(\frac{b(x, \tau)}{a(x, \tau)}\right)^{1/4} \tag{4.7}$$

The derivation of the particular solution $Q^*$ of the nonhomogeneous equation of the second order is obvious, when the fundamental system of solutions of the corresponding homogeneous equation is known [8].

Combining the particular solution and the exponentially increasing and decreasing solutions, after transformation to the old variables, we obtain a complicated expression which is analogous to the expression (1.10) of [5]. Thus, we have set up the asymptotic solution of a system with distributed parameters [see the problems (4.3), (4.4)] as a system with one conditional degree of freedom. Earlier (in Section 2) the system with distributed parameters was replaced by a system with one slowly varying degree of freedom. Then, the same complete system is replaced by a system with one slowly varying degree of freedom, but on a variable interval. The one degree of freedom being considered can thus be called only conditional, since one degree of freedom on a variable interval does not agree with the usual notion about a degree of freedom of a certain oscillatory system.

The solution thus set up enables us to proceed to the approximation of a system with an infinite number of degrees of freedom by a system with a finite number of degrees of freedom. However, now, in contrast to Section 3, the decisive quantity is the length of the wave of the loss of stability, and not the number of zeros of the mode of loss of stability. This is connected by the fact that earlier only sufficiently smooth functions were considered. Generally speaking, the function N(x, t) can be a discontinuous function.

Among the modes of loss of stability which approximate the original system there certainly must be modes of loss of stability with a local wavelength $L = L(x)$ which corresponds to the maximum of the index of the exponentially increasing solution (4.6) at any instant of time.

We note that the experimental results from [6, 7] fairly well agree with the solution thus set up, and namely under constant loading of a homogeneous bar the zeros of the deflection function are only a little displaced, while the amplitude distribution has an exponential character. For infinitely high velocity of stress propagation, the bar can be considered as a system with one degree of freedom; here, the maximum of the exponential index corresponds to this degree of freedom (see Section 2 and [1]).

## 5. On the Critical Time and the Critical Intensity of Loading in the Buckling Process of Bars

Above (see Sections 2 and 4), we have carried out an asymptotic analysis of buckling, when a system with distributed parameters was replaced by a system with one degree of freedom. At each instant of time under aperiodic intensive loading, we selected the mode of loss of stability which has the highest growth rate [see (2.1), (4.6)]. The displacements (the amplitudes of normal deflection) of the system are seen to be overstated in comparison with those actually taking place. But in such a case, the simple analytical relationships (2.1) and (4.6) can naturally be used to obtain estimates of the critical time and the critical intensity of loading in the buckling process. Here, and this is particularly important, the estimate of this time and this intensity will be an estimate from below. However, it must be emphasized that the expressions (2.1) and (4.6) were derived with assumption of active loading, i.e., there is only an intensive compressive load acting along the bar.

Under a critical time or a critical intensity of loading, we understand the lower estimate of them, if the behavior of the entire system at any instant of time (including instants of time after removal of the load) is determined by the active loading portion.

The critical time $t_*$ in the buckling process, or the critical loading intensity $\eta_*$ are determined from the relationships, in which the chosen decisive quantity is the maximum of the deflection

$$\max|w(x, \eta, t_*)| = w_*, \quad \max|w(x, \eta_*, t_0)| = w_* \tag{5.1}$$

or the magnification factor (see [3])

$$\frac{\max|w(x, \eta, t_*)|}{\max|w_0(x)|} = w_{**}, \quad \frac{\max|w(x, \eta_*, t_0)|}{\max|w_0(x)|} = w_{**} \tag{5.2}$$

Here, $t_*$ is the critical time, $t_0$ is a certain fixed instant of time, $\eta_*$ is the critical intensity of loading, $w_*$ is the maximum permissible deflection of the elastic system for the given disturbances which take place under intensive loading, and $w_{**}$ is the critical magnification factor. This factor is the ratio of the maximum value of the additional deflection at the final instant of time $|w(x, \eta, t)|$ to the maximum value of the initial deflection $|w_0(x)|$; $w_0(x)$ is a function which characterizes the initial imperfections of the bar.

When using the first expression, the estimates of $t_*$ and $\eta_*$ are obtained as the final ones, when an initial deflection of the bar is absent $w_0(x) \equiv 0$, but certain disturbances during loading are present (for example, a small load perpendicular to the axis of the bar). When $w_0(x) \equiv 0$, the second expressions (5.2) must not be used. After certain obvious transformations and a renotation, (5.2) is reduced to (5.1), if $w_0(x) \not\equiv 0$. Subsequently, the relationships (5.1) are considered. The criteria proposed follow from the definition adopted in the engineering theory of stability of motion on a finite time interval $t \in [0, t_*]$ or $t \in [0, t_0]$.

The relationships (5.1) can be considerably simplified, if we bear in mind the order of the quantities in the relationships (2.1) and (4.6)

$$\eta \gg 1, \; z_0(x, t) = O(1), \; z_0(x, \tau) = O(1), \; Z_0(t, x) = O(1) \tag{5.3}$$

$$Z_0(\tau, x) = O(1),\ \mu_{22}(t, x) = O(1),\ \mu_{22}(\tau, x) = O(1)$$

Therefore, instead of the functions $z_0$, $Z_0$, and $\mu_{22}$ in the simplified relationships (5.1), we use the quantities

$$z_{00} = \max |z_0(x, \xi_1)|,\quad Z_{00} = \max |Z_0(\xi_1, x)|,\quad \mu_{22}^{\circ} = \max |\mu_{22}(\xi_1, x)| \tag{5.4}$$

Here, $\xi_1 = t$, when the velocity of propagation of the longitudinal disturbances is taken as infinite; $\xi_1 = \tau$, when the velocity of propagation of the longitudinal disturbances is finite.

If we use the constants $z_0$, $Z_{00}$, and $\mu_{22}^{\circ}$ instead of the functions in the expressions (2.1) and (4.6), then the relationships (5.1) as a rule can be solved for the critical parameters $t_*$ and $\eta_*$ (it is understood that in (2.1) and (4.6) $\max |\sin \xi| = 1$).

We mention the difference between the method proposed here for the determination of critical time and critical intensity according to the expressions following from the relationships (5.1), and the methods of [3]. D. L. Anderson and H. E. Lindberg propose to calculate the magnification factor for all modes of loss of stability, and these modes are chosen, generally speaking, without sufficient justification (see the expressions (5) and (6) in [3]); then, from the maximum of the magnification factor for certain modes they propose to judge the behavior of the entire system. Here, however, the mode of loss of stability is chosen in a special manner – the rate of growth of the deflections is deliberately increased, in order to obtain the lower estimate for the critical time $t_*$ and the critical intensity $\eta_*$ under active loading. In addition, the practical calculations according to the method proposed are simpler than the calculation of the entire magnification curve. In a particular case, when the loading is constant, the critical parameters are calculated particularly simply. In essence, the same results are obtained as in [1], since the mode is selected which "corresponds to the largest coefficient in the exponential index of the function of time" (see [1], p. 780).

Example. Let the deflections of the bar, when the velocity of propagation of the longitudinal disturbances is taken as infinite, be satisfactorily described by the expression (2.1). Then, bearing in mind (5.3) and (5.4), we have

$$\eta^2 t \mu_{22}^0 = \ln w_* - \ln C_1 - \ln z_{00} - \ln Z_{00} \tag{5.5}$$

Let the difference ($\ln w_* - \ln C_1$) be not close to zero. We drop the second-degree terms from (5.5) [see (5.3)]. For the critical time $t_*$, and the critical intensity of loading $\eta_*$, we obtain the simple expressions

$$t_* = \frac{\ln w_* - \ln C_1}{\eta^2 \mu_{22}^{\circ}},\quad \eta_* = \left(\frac{\ln w_* - \ln C_1}{\mu_{22}^{\circ} t_0}\right)^{1/2} \tag{5.6}$$

The simple relationships (5.6) thus obtained, are very stable in respect to errors which are possible when determining $w_*$ and $C_1$.

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